Is particle creation by the gravitational field consistent with energy conservation?

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Abstract

If particle creation is described by a Bogoliubov transformation, then, in the Heisenberg picture, the raising and lowering operators are time dependent. On the other hand, this time dependence is not consistent with field equations and the conservation of the stress-energy tensor. Possible physical interpretations and resolutions of this inconsistency are discussed.

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It is widely believed that background gravitational field can cause production of particles [1]. However, the theoretical framework that describes this hypothetical effect, based on Bogoliubov transformation, is still far from being free of fundamental and conceptual problems. One of the problems is how particle creation from the vacuum can be consistent with the conservation of energy. It is often argued that creation of particles causes a backreaction on the background metric in such a way that energy is conserved. However, this conjecture has never been proved rigorously. In this letter we argue that the backreaction cannot solve the energy problem if the description of particle creation is based on Bogoliubov transformation. More precisely, we show that continuous change of the average number of particles is inconsistent with the local conservation of the stress-energy tensor.

The background metric is described by the semi-classical Einstein equation

$$\frac{1}{2}g_{\mu\nu}R - R_{\mu\nu} = 8\pi G\langle\psi|T_{\mu\nu}|\psi\rangle \ . \tag{1}$$

We assume that the backreaction caused by all physical processes, including a possible particle production, is included in (1). We only exclude physical processes related to a

collapse of the quantum state $|\psi\rangle$, because the semi-classical equation is not consistent when the collapse is taken into account [2]. For simplicity, we assume that (1) refers to quantities renormalized such that other possible geometrical terms on the left-hand side [1] are not present. Since, by assumption, the backreaction is included exactly, equation (1) must be exact. The covariant divergence of the left-hand side vanishes identically, so the covariant divergence of the right-hand side must vanish too. We work in the Heisenberg picture (in which the state $|\psi\rangle$ is constant), which implies that the operator equation

$$D^{\mu}T_{\mu\nu} = 0 \tag{2}$$

must be valid exactly. For simplicity, we assume that $T_{\mu\nu} \equiv T_{\mu\nu}(\phi)$ is determined by the action of a hermitian scalar field ϕ coupled only to the exact background metric $g_{\mu\nu}$. Therefore, the corresponding curved space-time Klein-Gordon equation

$$(D^{\mu}\partial_{\mu} + m^2 + \xi R)\phi = 0 \tag{3}$$

should be valid exactly.

Let $\Sigma(t)$ denote some foliation of space-time into Cauchy spacelike hypersurfaces. The coordinates $x = (t, \mathbf{x})$ are chosen such that $t =$ constant on Σ. We want to show that if the average number of particles at Σ changes continuously with time t, then equations (2) and (3) are not satisfied.

The field ϕ can be expanded as

$$\phi(x) = \sum_k a_k f_k(x) + a_k^{\dagger} f_k^*(x) \; . \tag{4}$$

Here $f_k(x)$ are solutions of (3) such that they are positive-freqeuency solutions at some initial time t_0. We introduce the vacuum $|0\rangle$ as a state with the property $a_k|0\rangle = 0$. This state has zero particles at t_0. A general state $|\psi\rangle$ is constructed by acting on $|0\rangle$ with the operators $a_k^{\dagger}$. To find the average number of particles at some other time $t > t_0$, we introduce a new set of functions $u_l(x)$ that satisfy (3) and are positive-freqeuency solutions at t. Instead of (4), we can use the expansion

$$\phi(x) = \sum_l A_l u_l(x) + A_l^{\dagger} u_l^*(x) \; . \tag{5}$$

The functions u_l satisfy the relations

$$\begin{aligned} (u_l, u_{l'}) = -(u_l^*, u_{l'}^*) = \delta_{ll'} \; , \\ (u_l, u_{l'}^*) = (u_l^*, u_{l'}) = 0 \; , \end{aligned} \tag{6}$$

where

$$(\varphi, \chi) = i \int_{\Sigma} d\Sigma^{\mu} \varphi^* \overleftrightarrow{\partial_{\mu}} \chi \; . \tag{7}$$

When $\varphi(x)$ and $\chi(x)$ are solutions of (3), then the scalar product (7) does not depend on Σ. The relations analogous to (6) are also valid for f_k. The two sets of functions are related by the Bogoliubov transformation

$$u_l(x) = \sum_k \alpha_{lk} f_k(x) + \beta_{lk} f_k^*(x) \; , \tag{8}$$

where

$$\alpha_{lk} = (f_k, u_l) \ , \quad \beta_{lk} = -(f_k^*, u_l) \ . \tag{9}$$

From the requirement that (4) and (5) should be equal, one finds

$$A_l = \sum_k \alpha_{lk}^* a_k - \beta_{lk}^* a_k^\dagger \ . \tag{10}$$

Equation (10) is sufficient to express the average number of particles in a state $|\psi\rangle$ at the time t. However, to see the relevance of the stress-energy tensor explicitly, we use a formalism which is equivalent, but more complicated. The time evolution is generated by the Hamiltonian

$$H(t) = \int_{\Sigma(t)} d\Sigma^\mu n^\nu T_{\mu\nu}(\phi) \ , \tag{11}$$

where n^ν is a unit vector normal to $\Sigma(t)$. The time dependence of the left-hand side is a consequence of the time dependence of the metric $g_{\mu\nu}$. One could also add the gravitational contribution to the right-hand side of (11). However, this contribution depends only on the c-numbers $g_{\mu\nu}$, not on the operators ϕ, so it is physically irrelevant to the time evolution of the operators that describe matter. The time evolution of the particle-number operator is described by

$$N(t) = U(t, t_0) N(t_0) U^\dagger(t, t_0) \ , \tag{12}$$

where $N(t_0) = \sum_k a_k^\dagger a_k$. The unitary operator U satisfies the Schrödinger equation

$$i\frac{d}{dt} U(t, t_0) = H(t) U(t, t_0) \ . \tag{13}$$

Putting (5) into the expression for $T_{\mu\nu}(\phi)$ in (11), equations (12) and (13) give [3]

$$N(t) = \sum_l A_l^\dagger A_l \ . \tag{14}$$

Of course, we could obtain this directly, without using (11), (12) and (13).

The formalism presented above is not new. However, in order to understand the discussion that follows, it was important to explicitly explain the crucial steps in the derivation of (14). The problem is that A_l in (5) are constant operators, so (14) implies $dN(t)/dt = 0$. In other words, the average number of particles does not continuously change with time:

$$\frac{d}{dt} \langle\psi|N(t)|\psi\rangle = 0 \ . \tag{15}$$

To avoid this difficulty, one expects that (14) should be replaced by an expression of the form

$$N(t) = \sum_l A_l^\dagger(t) A_l(t) \ . \tag{16}$$

It is not difficult to understand the origin of this extra time dependence. To describe the continuous creation of particles, we need a new set of functions $u_l(x)$ for *each* time t. This means that the modes u_l possess an extra continuous time dependence, i.e. they

become functions of the form $u_l(x;t)$. These functions do not satisfy (3). However, the functions $u_l(x;\tau)$ satisfy (3), provided that τ is kept fixed when the derivative ∂_μ acts on u_l. For $\varphi(x;t)$, $\chi(x;t)$ being two arbitrary functions with such an extra t-dependence, we define the scalar product as

$$(\varphi,\chi)_t \equiv i\int_{\Sigma(t)} d\Sigma^\mu \varphi^*(x;\tau)\overleftrightarrow{\partial_\mu}\chi(x;\tau)\,|_{\tau=t}\;, \tag{17}$$

where τ is kept fixed when the derivative acts. In general, this scalar product depends on t, even if $\varphi(x;\tau)$ and $\chi(x;\tau)$ satisfy (3) when τ is kept fixed. The functions $u_l(x;t)$ satisfy relations (6) with respect to the scalar product (17). However, the Bogoliubov coefficients (9), calculated using the scalar product (17), become t-dependent. Therefore, (10) should be replaced by

$$A_l(t)=\sum_k \alpha^*_{lk}(t)a_k-\beta^*_{lk}(t)a^\dagger_k\;. \tag{18}$$

This relation can be derived from the requirement that (4) should be equal to

$$\phi(x)=\sum_l A_l(t)u_l(x;t)+A^\dagger_l(t)u^*_l(x;t)\;, \tag{19}$$

or

$$\phi(x)=\sum_l A_l(t)u_l(x)+A^\dagger_l(t)u^*_l(x)\;, \tag{20}$$

provided that the t-dependence of $A_l(t)$ is treated as an extra t-dependence in (17). In other words, (18) is valid if the relation $(u_{l'},A_l(t)u_l)_t=A_l(t)\,(u_{l'},u_l)_t$ (as well as other similar relations) is valid.

Now it seems that we have a consistent derivation of the extra time dependence in (16). However, it is not consistent. The fields (19) and (20) do *not* satisfy the Klein-Gordon equation (3). Therefore, (19) and (20) cannot be equal to (4). This implies that equation (18), which expresses the equality of (4), (19) and (20), cannot be consistent either. This is, indeed, true because the scalar product (17) is inconsistently defined. Namely, the "extra" t-dependence cannot really be distinguished from the "regular" t-dependence, because one can always write $\varphi(x;t)\equiv\tilde{\varphi}(x)$. On the other hand, to obtain a relation similar to (16) by using (11), (12) and (13), one needs to put (19) or (20) into the expression for $T_{\mu\nu}(\phi)$. Since (19) and (20) do not satisfy (3), this $T_{\mu\nu}$ does not satisfy (2). Since the expression for $T_{\mu\nu}(\phi)$ involves the time derivatives $\dot{\phi}$, the resulting expression for $N(t)$ will take the form

$$N(t)=\sum_l A^\dagger_l(t)A_l(t)+\mathcal{N}(\dot{A}_l(t),\dot{u}_l(\tau,\mathbf{x};t)|_{\tau=t}). \tag{21}$$

The extra term $\mathcal{N}$ obtained using (19) is not the same as that obtained using (20). This term is negligible if we assume that the change of the average number of particles is slow, i.e. that $\dot{A}_l(t)\approx 0$, $\dot{u}_l(\tau,\mathbf{x};t)|_{\tau=t}\approx 0$. If we take this approximation, then (3) and (2) are approximately valid. However, although the particle production can be slow, the total number of produced particles can be significant after a long time. Similarly, the total produced energy can be significant after a long time.

Our results can be summarized as follows: If particle creation is described by a Bogoliubov transformation, then, in the Heisenberg picture, the raising and lowering operators are time dependent. On the other hand, this time dependence is not consistent with the field equations and the conservation of the stress-energy tensor. Below we discuss several possible approaches to the resolution of this problem and show that none of them is completely satisfactory.

One possibility is to define the average of the stress-energy tensor in an independent way, without using (19) or (20). Indeed, the stress-energy tensor is often defined using a formalism based on the Schwinger-DeWitt representation of the Green function [1]. The stress-energy tensor defined in this way is automatically conserved, but is *not* represented as $\langle\psi|T_{\mu\nu}|\psi\rangle$ for some operator $T_{\mu\nu}$. Such an approach leads to an argument that particle production is consistent with the local conservation of the stress-energy tensor [1]. However, we do not find this argument satisfactory, because the formalism in which the average number of particles is described by an operator, whereas the average stress-energy tensor is not described by an operator, does not seem to be consistent. Moreover, if one does not define the operator $T_{\mu\nu}$, then one cannot describe the particle production by the formalism based on (11). Finally, it is not clear in such an approach whether the Klein-Gordon equation (3) is satisfied.

Another possibility (that can be combined with the possibility above) is that a well-defined operator $N(t)$ simply does not exist, even when the foliation $\Sigma(t)$ is chosen. Instead, the number of particles should be defined operationally, by the response of a "particle" detector of the Unruh-DeWitt type [4, 5]. Such detectors need a long time to measure the number of particles in a reliable way. The average number of particles at a given time t loses its meaning. However, since all other observables in quantum mechanics can be represented by well-defined hermitian operators that evolve continuously with time and do not require a model of the correspoding detector, it is not clear why the number of particles should be an exception.

At this point it is instructive to compare the concepts of energy and particle number in flat space-time. In this case, both quantities are conserved. On the other hand, both quantities obey certain approximate uncertainty relations [6]

$$\begin{aligned} \Delta E \Delta t &\geq 1 \, , \\ \Delta N \Delta t &\geq m^{-1} \, , \end{aligned} \tag{22}$$

where m is the mass of the particle and Δt is the time interval during which the measurement is performed. These relations cannot be derived from some fundamental quantum principles. They merely express the uncertainties related to typical methods of measurement. Actually, it is, in principle, possible to measure energy with an arbitrary accuracy inside an arbitrarily small time interval [7]. There is no reason why this would not be the case for the number of particles as well. The uncertainty relations do not imply that the operators $H(t)$ and $N(t)$ are not well defined, even when they are not conserved.

Our discussion suggests a new approach to the resolution of the problem of inconsistency between particle creation and energy conservation. In this approach, equation (15) is interpreted as conservation of the particle number when the classical gravitational interaction described by (3) is the only interaction. It seems that in such an approach

one has to reject a common belief that the definition of particles should be closely related to the definition of positive frequencies. We shall study such a possibility in more detail elsewhere.

A close relationship between the non-conservation of energy and particle number can also be seen in the following way: Assume that space-time is flat at t_0 and t, but not at the intermediate times. In this case, the non-conservation of energy is obvious from the relations

$$\begin{aligned} H(t_0) &= \textstyle\sum_k \omega_k (a_k^\dagger a_k + \frac{1}{2}) \ , \\ H(t) &= \textstyle\sum_k \omega_k (A_k^\dagger A_k + \frac{1}{2}) \ . \end{aligned} \tag{23}$$

For example, $\langle 0|H(t)|0\rangle - \langle 0|H(t_0)|0\rangle = \sum_k \sum_{k'} \omega_k |\beta_{kk'}|^2/2$. The fact that the energy should be conserved suggests that certain operators should be renormalized such that the renormalized Hamiltonian is conserved. A formalism which renormalizes the Hamiltonian should also renormalize the number of particles. The equations given above suggest the following renormalization of the raising and lowering operators:

$$A_k^{\rm ren} = A_k - \sum_{k'} [(\alpha^*_{kk'} - \delta_{kk'}) a_{k'} - \beta^*_{kk'} a^\dagger_{k'}] \ . \tag{24}$$

Since $A_k^{\rm ren} = a_k$, such a renormalization leads to $H^{\rm ren}(t) = H(t_0)$. A necessary consequence of such a renormalization is that there is no particle production because $N^{\rm ren}(t) = N(t_0)$. The problem with this heuristic argument against particle production is that (24) is obtained in an ad hoc way. One needs a derivation that starts from more fundamental principles.

There is no doubt that the usual concept of particles that emerges from free fields in Minkowski space-time should be modified significantly when the generalization to arbitrary space-time is considered. Our analysis demonstrates that existing achievements in this field are far from being completely satisfying. Further investigation is needed in order to formulate a closed and consistent theory.

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References

[1] N. D. Birrell and P. C. W. Davies, Quantum fields in curved space (Cambridge Press, NY, 1982).

[2] D. N. Page and C. D. Geilker, Phys. Rev. Lett. 47 (1981) 979.

[3] L. P. Grishchuk and Y. V. Sidorov, Phys. Rev. D 42 (1990) 3413.

[4] W. G. Unruh, Phys. Rev. D 14 (1976) 870.

[5] B. S. DeWitt, in: General Relativity: An Einstein Centenary Survey, eds. S. W. Hawking and W. Israel (Cambridge University Press, 1979).

[6] L. Parker, Phys. Rev. 183 (1969) 1057.

[7] Y. Aharonov and D. Bohm, Phys. Rev. 122 (1961) 1649.